# ABSTRACT

DEPARTMENT OF PHYSICS

MIXON, MELODY D. B.S. CLARK COLLEGE, 1990

## APPROXIMATE SOLUTIONS TO THE ANTI–SYMMETRIC QUADRATIC SPRING SYSTEM

Advisor: Professor Ronald E. Mickens

Thesis dated March 1992

We investigate the construction of analytic approximations to the solutions of the nonlinear anti–symmetric quadratic oscillator. The procedure used is based on the method of generalized harmonic balance as formulated by Mickens. We prove that all solutions to the differential equation are bounded and periodic. This result is based upon the use of a correct definition of the absolute value of a variable and the use of this definition to calculate its derivative.

APPROXIMATE SOLUTIONS TO THE

ANTI–SYMMETRIC QUADRATIC SPRING SYSTEM

A THESIS

SUBMITTED TO THE FACULTY OF CLARK ATLANTA UNIVERSITY

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR

THE DEGREE OF MASTER OF SCIENCE

BY

MELODY D. MIXON

DEPARTMENT OF PHYSICS

ATLANTA, GEORGIA

MARCH 1992

# NOTICE TO BORROWERS

All dissertations and theses deposited in the Atlanta University Center R. W. Woodruff Library must be used only in accordance with the stipulation prescribed by the author in the preceding statment.

The author of this thesis/dissertation is:


Name: Melody D. Mixon

Street Address: 4070 Montego Bay Drive

City, State and Zip: College Park, Georgia 30349


The director of this thesis/dissertation is:

Professor: Dr. Ronald E. Mickens

Department: Physics

School: Arts and Sciences
Clark Atlanta University

Office Telephone: (404) 880-8797

Users of this thesis/dissertation not regularly enrolled as students of the Atlanta University Center are required to attest acceptance of the preceding stipulations by signing below. Libraries borrowing this thesis/dissertation for use of patrons are required to see that each user records here the information requested.

| NAME OF USER | ADDRESS | DATE | TYPE OF USE |
|---|---|---|---|
| | | | |
| | | | |
| | | | |
| | | | |

## ACKNOWLEDGEMENTS

I would like to thank my thesis advisor, Professor Ronald E. Mickens, for his assistance and guidance throughout this project. I extend a special thanks to my parents for their love and continued support in my efforts to further my education. Last, but not least, I thank Jesus Christ: "I can do all things through Christ who strengthens me."

My research efforts were supported by NASA Grant NAG–1–410.

# TABLE OF CONTENTS

Page

Acknowledgements ii

Table of Contents iii

CHAPTER ONE: INTRODUCTION 1

1.1 Statement of the Problem 1

1.2 Summary of Thesis 1

1.3 Quadratic Nonlinearities 2

1.4 Outline of Thesis 3

CHAPTER TWO: BACKGROUND TOPICS 4

2.1 Derivative of $|y|$ 4

2.2 Energy Principle 4

2.3 Harmonic Balance 5

2.4 Fourier Series 7

2.5 Fourier Coefficients of $|\cos\theta|$ 7

CHAPTER THREE: CALCULATIONS 9

3.1 The Form $y = A\cos\theta/(1 + B\cos 2\theta)$ 9

3.2 The Form $y = A_1\cos\theta + B_1\cos 3\theta$ 11

CHAPTER FOUR: DISCUSSION 14

4.1 Summary and Discussion 14

4.2 Extensions of Research 17

REFERENCES 18

# CHAPTER ONE

# INTRODUCTION

## 1.1. Statement of the Problem

The paper of Mickens and Ramadhani[1] applied the perturbation method of Krylov–Bogoliubov–Mitropolsky to calculate an analytic approximation to the conservative oscillator

$$\ddot{y} + y + \epsilon |y| y = 0, \qquad 0 < \epsilon \ll 1, \tag{1.1}$$

where $\epsilon$ is a small parameter. These calculations covered more than thirty notebook pages. The huge amount of algebraic work is related to the fact that the absolute value, $|y|$, appears in the nonlinear term.

The problem investigated in this thesis is to apply the method of harmonic balance[2] to the related nonlinear equation

$$\ddot{y} + |y| y = 0. \tag{1.2}$$

This equation does not contain a linear term in $y$, consequently, none of the usual methods of perturbation can be used to determine approximations to its solutions. A major advantage of the harmonic balance procedure is that it can be applied to problems for which the nonlinearities are large. Also, there is no restriction, for conservative systems, on the size of any "small parameters" such as $\epsilon$.

## 1.2. Summary of Thesis

Using the method of generalized harmonic balance[2], the following form was assumed for an approximation to the solution of Eq. (1.2)

$$y(t) = \frac{A \cos \theta}{1 + B \cos 2\theta}, \qquad \theta = \omega t, \tag{1.3}$$

where $(A, B, \omega)$ are unknown amplitudes and the angular frequency. Selecting the initial conditions

$$y(0) = y_0 = \text{given}, \qquad \dot{y}(0) = 0, \tag{1.4}$$

gives the following results for $(A, B, \omega)$:

$$A \simeq (0.946) y_0, \tag{1.5}$$

$$B \simeq -0.054, \tag{1.6}$$

$$\omega \simeq (0.908)\sqrt{y_0}. \tag{1.7}$$

### 1.3. Quadratic Nonlinearities

The best known example of a nonlinear oscillator having a quadratic term containing an absolute value expression is the Lewis oscillator[3,4]

$$\ddot{y} + y = \epsilon(1 - |y|)\dot{y}, \qquad 0 < \epsilon \ll 1. \tag{1.8}$$

This equation has a single stable limit–cycle (i.e., periodic solution). Consequently, the solutions to Eq. (1.8) have many of the features associated with the well–known van der Pol differential equation[5]

$$\ddot{y} + y = \epsilon(1 - y^2)\dot{y}, \qquad 0 < \epsilon \ll 1. \tag{1.9}$$

This is not surprising since the two functions

$$f_1(y) = 1 - |y|, \tag{1.10}$$

$$f_2(y) = 1 - y^2, \tag{1.11}$$

have the same topological properties and, from general mathematical considerations, must give rise to solutions having the same topological structure.[4,6]

Another interest in the study of oscillatory systems with quadratic non-linear terms, of the form $|y|y$, is that this expression is non–analytic at $y = 0$ due to the appearance of the absolute value $|y|$. One consequence is that the usual expansion in $\epsilon$ perturbation series methods[5] cannot be applied to this situation.[1] However, averaging procedures do work as illustrated by the paper of Mickens and Ramadhani[1]. The major problem with averaging procedures is the huge amount of algebraic work required to obtain an answer.

## 1.4. Outline of Thesis

Chapter Two presents a variety of background material in summary form. These include a proof that all the solutions to Eq. (1.2) are bounded and periodic; the definition of the Fourier coefficients for a periodic function; and the Fourier expansions for $|\cos\theta|$ and $|\cos\theta|\cos\theta$. Chapter Three gives the details of the calculations for the determination of the approximate solutions to Eq. (1.2). Finally, our results and possible future extensions are discussed in Chapter Four.

# CHAPTER TWO

# BACKGROUND TOPICS

## 2.1. Derivative of $|y|$

The absolute value function, $|y|$, appears in Eq. (1.2). It is of interest to select a definition for this function that allows a direct means of obtaining its derivative. In particular this is needed to show that Eq. (1.2) follows from the energy principle given in the next section.

We define $f(y) = |y|$ as follows

$$f(y) = |y| \equiv \sqrt{y^2}. \tag{2.1}$$

This definition allows the derivative of $f(y)$ to be found by directly applying the usual rules of calculus. Doing this gives

$$\frac{df}{dy} = \left(\frac{1}{2}\right)(y^2)^{\frac{1}{2}-1}(2y) = \frac{y}{\sqrt{y^2}} = \frac{y}{|y|}, \tag{2.2}$$

or

$$\frac{df}{dy} = \operatorname{sign}(y), \tag{2.3}$$

where the "sign" function is

$$\operatorname{sign}(y) = \begin{cases} +1, & \text{for } y > 0; \\ -1, & \text{for } y < 0. \end{cases} \tag{2.4}$$

## 2.2. Energy Principle

The differential equation (1.2) has the following "first integral" or energy conservation law

$$\frac{(\dot{y})^2}{2} + V(y) = E > 0, \tag{2.5}$$

where the constant $E$ is the total energy and the potential energy function $V(y)$ is

$$V(y) = \frac{|y|y^2}{3}. \tag{2.6}$$

To prove that Eq. (2.5) is a first integral, one need only take its derivative with respect to time and then use the result given by Eq. (2.2). Doing this gives

$$\dot{y}\ddot{y} + |y|y\dot{y} = 0, \tag{2.7}$$

and

$$\ddot{y} + |y|y = 0, \qquad \dot{y} \not\equiv 0. \tag{2.8}$$

This non–negativity of both terms in Eq. (2.5) implies that all solutions to this equation are periodic. In other words, in the phase–plane, with variables $(y, \dot{y})$, the solution curves

$$\dot{y} = \dot{y}(y) \tag{2.9}$$

are all closed. (This result follows directly from Lemma 2.2, page 15, of Verhulst.[6])

### 2.3. Harmonic Balance

The method of harmonic balance can be used to construct analytic approximations to the periodic solutions of differential equations. The basis of the method rests on the fact that the equations of interest have periodic solutions whose Fourier coefficients decrease rapidly. The paper of Mickens[2] gives conditions that the differential equation must satisfy such that harmonic balance applies. In simplest form, an assumed solution

$$y = A\cos\omega t, \tag{2.10}$$

is substituted into the nonlinear differential equation

$$F(\ddot{y}, \dot{y}, y, t) = 0, \tag{2.11}$$

where $F$ can be a nonlinear function of its arguments. After suitable mathematical manipulation, the following result is obtained

$$g_1(A, \omega) \cos \omega t + g_2(A, \omega) \sin \omega t + \text{higher harmonics} = 0, \tag{2.12}$$

where $g_1$ and $g_2$ are known functions of the amplitude $A$ and frequency $\omega$. The essence of the method is to ignore the higher harmonics and, consequently, set the coefficients of $\cos \omega t$ and $\sin \omega t$ equal to zero, i.e.,

$$g_1(A, \omega) = 0, \qquad g_2(A, \omega) = 0. \tag{2.13}$$

For "conservative" systems, $g_2(A, \omega)$ does not occur and there is only one equation which can be solved to relate the angular frequency to the amplitude $A$. For non–conservative systems, we have two equations to be solved to give the values of limit–cyles and limit–points. Multi–solutions of these equations correspond to multi–periodic solutions of the original differential equation.

The method of simple harmonic balance, as given by Eq. (2.10), can be generalized.[2] For example, the second–order approximation takes one of the following forms

$$y(t) = A_1 \cos \theta + B_1 \cos 3\theta, \tag{2.14}$$

$$y(t) = \frac{A \cos \theta}{1 + B \cos 2\theta}, \tag{2.15}$$

where $(A_1, B_1, A, B)$ are the unknown amplitudes and $\omega$ is the angular frequency. The variable $\theta$ is

$$\theta = \omega t. \tag{2.16}$$

## 2.4. Fourier Series

Consider a periodic function $x(\theta) = x(\theta + 2\pi)$. Its formal Fourier expansion is

$$x(\theta) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty}[a_k \cos(k\theta) + b_k \sin(k\theta)], \tag{2.17}$$

where the Fourier coefficients are given by the expressions

$$a_k = \left(\frac{1}{\pi}\right) \int_{-\pi}^{\pi} x(\theta) \cos(k\theta) d\theta, \tag{2.18}$$

$$b_k = \left(\frac{1}{\pi}\right) \int_{-\pi}^{\pi} x(\theta) \sin(k\theta) d\theta. \tag{2.19}$$

If $x(\theta)$ is single–valued, has a finite number of maximum and minimum values, and is continuous, then the Fourier expansion of Eq. (2.17) converges to $x(\theta)$.

Note that for our problem, the periodic solution, $y(\theta)$, where $\theta = \omega t$, has at every point a third derivative.

## 2.5. Fourier Coefficients of $|\cos\theta|$

From Eqs. (2.18) and (2.19), the following results are obtained for the Fourier coefficients, $a_k$ and $b_k$, of the function $|\cos\theta|$:

$$\begin{aligned} a_k &= \left(\frac{1}{\pi}\right) \int_{-\pi}^{\pi} |\cos\theta| \cos(k\theta) d\theta \\ &= \left(\frac{2}{\pi}\right) \int_{0}^{\pi} |\cos\theta| \cos(k\theta) d\theta \\ &= \left(\frac{2}{\pi}\right) \left[\int_{0}^{\pi/2} \cos\theta \cos(k\theta) d\theta - \int_{\pi/2}^{\pi} \cos\theta \cos(k\theta) d\theta\right]. \end{aligned} \tag{2.20}$$

Evaluation of the integrals gives

$$\text{(2.21)} \qquad \begin{aligned} a_0 &= \frac{4}{\pi}, \qquad a_1 = 0, \\ a_k &= \left(\frac{-4}{\pi}\right)\frac{\cos\left(\frac{k\pi}{2}\right)}{(n-1)(n+1)}, \qquad k = 2, 3, 4, \ldots. \end{aligned}$$

Likewise,

$$\text{(2.22)} \qquad b_k = \left(\frac{1}{\pi}\right)\int_{-\pi}^{\pi} |\cos\theta| \sin(k\theta)d\theta = 0.$$

Therefore, we have

$$\text{(2.23)} \qquad \begin{aligned} |\cos\theta| = \left(\frac{4}{\pi}\right)\Bigg[\left(\frac{1}{2}\right) + \left(\frac{1}{3}\right)\cos 2\theta - \left(\frac{1}{15}\right)\cos 4\theta + \left(\frac{1}{35}\right)\cos 6\theta \\ - \left(\frac{1}{63}\right)\cos 8\theta + \left(\frac{1}{99}\right)\cos 10\theta + \cdots\Bigg]. \end{aligned}$$

# CHAPTER THREE

# CALCULATIONS

## 3.1. The Form $y = A\cos\theta/(1 + B\cos 2\theta)$

We will now calculate an approximation to the periodic solutions of the differential equation

$$\ddot{y} + |y|y = 0 \tag{3.1}$$

using the second approximation of the generalized harmonic balance method. The assumed solution takes the form[2]

$$y(t) = \frac{A\cos\theta}{1 + B\cos 2\theta}, \qquad \theta = \omega t. \tag{3.2}$$

The initial conditions are selected to be the following values

$$y(0) = y_0 = \text{given}, \qquad \dot{y}(0) = 0. \tag{3.3}$$

The unknown frequency $\omega$, and amplitudes $A$ and $B$, will be determined as functions of $y_0$, the initial position of the oscillator.[2,5] To obtain the necessary relations, Eq. (3.2) must be substituted into Eq. (3.1), the coefficients of $\cos\theta$ and $\cos 3\theta$ calculated, and then set equal to zero. This is what the method of harmonic balance requires. (See Section 2.3 and Mickens.[2])

A long, but, direct calculation using the result of Section 2.5 and the trigonometric relation

$$\cos\theta_1 \cos\theta_2 = \left(\frac{1}{2}\right)[\cos(\theta_2 - \theta_1) + \cos(\theta_2 + \theta_1)] \tag{3.4}$$

gives the following expressions:

(3.5)

$$(1 + B\cos 2\theta)^3|y|y = \left(\frac{8A^2}{3\pi}\right)\left\{\left[1 + \frac{3B}{5}\right]\cos\theta + \left(\frac{1}{5}\right)\left[1 + \frac{17B}{7}\right]\cos 3\theta + \text{HOH}\right\},$$

and

(3.6)

$$(1 + B\cos 2\theta)^3 \ddot{y} = -(\omega^2 A)\left\{\left[1 + B - \frac{11B}{2}\right]\cos\theta + 3B\left[\frac{3B}{4} - 1\right]\cos 3\theta + \mathrm{HOH}\right\},$$

where HOH stands for higher–order–harmonics. Substitution of Eqs. (3.5) and (3.6) into Eq. (3.1) and equating the coefficients of $\cos\theta$ and $\cos 3\theta$ equal to zero gives, respectively, the two equations:

(3.7)
$$\left(\frac{8A^2}{3\pi}\right)\left[1 + \frac{3B}{5}\right] = \omega^2 A\left[1 + B - \frac{11B^2}{2}\right],$$

(3.8)
$$\left(\frac{8A^2}{15\pi}\right)\left[1 + \frac{17B}{7}\right] = \omega^2 A\left[\frac{9B^2}{4} - 3B\right].$$

These are two equations for the three unknowns $A$, $B$ and $\omega$. To obtain an equation for $B$ divide Eq. (3.7) by Eq. (3.8). Doing this gives the cubic equation

(3.9)
$$\left(\frac{563}{28}\right)B^3 + \left(\frac{338}{28}\right)B^2 - \left(\frac{129}{7}\right)B - 1 = 0.$$

The solution of interest is that one which is real and small. Following the arguments of Mickens[2], a good approximation to this solution is given by the value

(3.10)
$$B \simeq -\frac{7}{129} \simeq -0.054.$$

Note that, as expected[2], $B$ is small, i.e.,

(3.11)
$$|B| \ll 1.$$

A consequence of this fact is that the denominator in the assumed functional form for the solution, Eq. (3.2) is always positive. Of additional interest is the observation that $B$ is a pure number that does not depend on the initial condition $y(0) = y_0$.

The angular frequency, $\omega$, can now be determined from Eq. (3.8). It is given by the expression

$$\omega^2 = \left(\frac{8}{15\pi}\right) \frac{\left[1 + \frac{17B}{7}\right] A}{\left[\frac{9B^2}{4} - 3B\right]}. \tag{3.12}$$

The substitution of the value for $B$, given by Eq. (3.10), gives the result

$$\omega \simeq 0.933\sqrt{A}. \tag{3.13}$$

We must now determine the amplitude $A$ in terms of the initial conditions given by Eq. (3.3). From Eq. (3.2), it follows that

$$y_0 = \frac{A}{1+B}. \tag{3.14}$$

Using the estimate for $B$, as expressed by Eq. (3.10), and solving for $A$ gives

$$A \simeq \left(\frac{122}{129}\right) y_0 \simeq (0.946) y_0, \tag{3.15}$$

and

$$\omega \simeq 0.908\sqrt{y_0}. \tag{3.16}$$

Putting all of these results back into our assumed solution, Eq. (3.2), we obtain the following expression for the periodic solutions of Eq. (3.1):

$$y(t) = \frac{(0.946) y_0 \cos(0.908\sqrt{y_0}t)}{1 - (0.054)\cos(1.816\sqrt{y_0}t)}. \tag{3.17}$$

### 3.2. The Form $y = A_1 \cos\theta + B_1 \cos 3\theta$

There is a second expression that gives a second–order harmonic balance solution to the differential equation of Eq. (3.1). It is

$$y(t) = A_1 \cos\theta + B_1 \cos 3\theta, \qquad \theta = \omega t, \tag{3.18}$$

where $A_1$, $B_1$, and $\omega$ are to be determined as functions of the initial conditions.

One way to proceed is to substitute Eq. (3.18) into Eq. (3.1), expand the resulting expression into a trigonometric polynomial, and set the coefficients of the $\cos\theta$ and $\cos 3\theta$ terms equal to zero. This would, as in the previous section, give two relations for the three unknowns $(A_1, B_1, \omega)$. One could then solve for $B_1$ and $\omega$ as functions of $A_1$, and then determine $A_1$ in terms of $y_0$. Hence, one would finally obtain everything as a function of $y_0$, i.e.,

$$A_1 = A_1(y_0), \qquad B_1 = B_1(y_0), \qquad \omega = \omega(y_0). \tag{3.19}$$

We have done this. However, the calculations are long and tedious.

An easier way to do this calculation is to start with Eq. (3.2) and use our knowledge of the smallness of $B$ to expand the denominator. Doing this gives

$$y = \frac{A\cos\theta}{1 + B\cos 2\theta} = A\cos\theta[1 - B\cos 2\theta + O(B^2)]. \tag{3.20}$$

Using the trigonometric relation of Eq. (3.4) allows us to write this as

$$y = A\left[1 - \frac{B}{2}\right]\cos\theta - \left(\frac{AB}{2}\right)\cos 3\theta + O(B^2). \tag{3.21}$$

Comparing Eqs. (3.18) and (3.21) gives

$$A_1 = A\left[1 - \frac{B}{2}\right], \qquad B_1 = -\left(\frac{AB}{2}\right). \tag{3.22}$$

Keeping $B$ fixed at the value $B = -7/129$, we can determine $A$ from the relation

$$y_0 = A - AB, \tag{3.23}$$

or

$$A \simeq (0.949)y_0. \tag{3.24}$$

This allows $\omega$, $A_1$ and $B_1$ to be calculated

$$\omega \simeq (0.933)\sqrt{A} = 0.909\sqrt{y_0}, \tag{3.25}$$

$$A_1 \simeq (0.975)y_0, \tag{3.26}$$

$$B_1 \simeq (0.026)y_0. \tag{3.27}$$

Thus to terms $O(B^2)$, with $B = -7/129$, the assumed second–order harmonic balance solution, having the form given by Eq. (3.18), is

$$y(t) = (0.975)y_0 \cos(0.909)\sqrt{y_0}t) + (0.026)y_0 \cos 3(0.909\sqrt{y_0}t). \tag{3.28}$$

# CHAPTER FOUR

# DISCUSSION

## 4.1. Summary and Discussion

The nonlinear oscillator differential equation

$$\ddot{y} + |y|y = 0, \tag{4.1}$$

is called an anti–symmetric quadratic oscillator because the force associative with it

$$F(y) = -|y|y, \tag{4.2}$$

is an odd function of $y$, i.e.,

$$F(-y) = -F(y). \tag{4.3}$$

The usual situation is a force function that can be written as

$$F_1(y) = -y^3, \tag{4.4}$$

or as

$$F_2(y) = -y^2. \tag{4.5}$$

The energy relations for the forces given by Eqs. (4.2), (4.4) and (4.5) are, respectively, the following forms

$$\frac{(\dot{y})^2}{2} + \frac{|y|y^2}{3} = E, \tag{4.2a}$$

$$\frac{(\dot{y})^2}{2} + \frac{y^4}{4} = E_1, \tag{4.4a}$$

$$(4.5a) \qquad \frac{(\dot{y})^2}{2} + \frac{y^3}{3} = E_2.$$

Note that in Eqs. (4.2a) and (4.4a), both terms on the left–sides of the respective equations are always non–negative. Thus, it may be concluded that in the respective phase–planes, $(y, \dot{y})$, all the solution curves are closed.[4,6] Hence, all the solutions to the original differential equations are periodic. However, the second term on the left–side of Eq. (4.5a) is negative if $y$ is negative and positive if $y$ is positve. For this case, no periodic solutions exist! Comparison of the second terms, on the left–sides, of Eqs. (4.2a) and (4.5a) shows that the replacement of a "$y$" by $|y|$ in Eq. (4.5a) completely changes the nature of the solutions.

It should also be indicated that the expression $y^3$ is an analytic (in the sense of the theory of complex variables) function of $y$, while the expression $|y|y^2$ is not an analytic function of $y$. As stated previously, the direct expansion methods of perturbation theory will not, in general, work for these situations. For our differential equation (4.1) there are two additional reasons why such methods cannot be applied. First, there is no linear term in $y$; second, no small parameter multiplies the nonlinear term. What we have demonstrated in this thesis is that the method of harmonic balance can be used to calculate approximations to the periodic solutions of nonlinear equations that contain non–analytic non–linearities. The procedure is simple to apply. The only possible difficulty might be the step that involves solving a high–order algebraic equation. However, the current existence of good computer software makes this a minor problem.

We have not discussed bounds on the errors in the calculated amplitudes and frequency. However, the general investigation of Mickens[2] indicates that these errors are of the order of 5% or less. Thus, the second–order method of harmonic balance provides an excellent approximation to the periodic solutions.

From Eq. (3.25), it is observed that the angular frqeuency, $\omega$, is proportional to the square–root of the initial position, $y(0) = y_0$, i.e.,

$$\omega \simeq 0.909\sqrt{y_0}. \tag{4.6}$$

This result is a consequence of the nonlinear term in Eq. (4.1) being quadratic. Also, the amplitude of the lowest harmonic (see $A$ in Eq. (3.2) or $A_1$ in Eq. (3.18)) is always proportional to $y_0$.

An advantage of the rational form

$$y = \frac{A\cos\theta}{1 + B\cos 2\theta}, \qquad \theta = \omega t, \tag{4.7}$$

as compared with the other approximation to the solution of Eq. (4.1), namely,

$$y = A_1 \cos\theta + B_1 \cos 3\theta, \tag{4.8}$$

is that Eq. (4.7) provides a functional relation that has harmonics of all the higher–orders included, i.e., Eq. (4.7) can be written as

$$y = A \sum_{k=1}^{\infty} a_{2k-1} \cos(2k-1)\omega t, \tag{4.9}$$

where the coefficients are functions only of $B$

$$a_{2k-1} = a_{2k-1}(B). \tag{4.10}$$

The inclusion of contributions from the higher–order harmonics means that the form of Eq. (4.7) provides a better approximation to the exact solution than does the expression of Eq. (4.8).

### 4.2. Extensions of Research

The research reported in this thesis can be extended in several directions. First, it might be of value to compare the approximate analytic solutions with accurate numerical solutions.

Second, a third-order rational form could be selected to determine an approximate solution. One possibility for $y$ is

$$y = \frac{A\cos\theta}{1 + B_1\cos 2\theta + B_2\cos 4\theta}, \qquad \theta = \omega t. \tag{4.11}$$

For this case, on substitution of $y$ into Eq. (4.1), the coefficients of the harmonics $\cos\theta$, $\cos 3\theta$ and $\cos 5\theta$ would be set equal to zero to obtain three algebraic equations for $A$, $B_1$, $B_2$ and $\omega$. The solutions for $B_1$ and $B_2$ would give pure numbers for their values. The angular frequency $\omega$ would be expressed in terms of $A$ which could then be given as a function of $y_0$.

Third, the differential equation

$$\ddot{y} + y + \epsilon|y|y = 0, \qquad 0 < \epsilon \ll 1, \tag{4.12}$$

which was investigated by Mickens and Ramadhani[1] using an averaging procedure could be looked at from the viewpoint of the method of generalized harmonic balance.

## REFERENCES


1. R. E. Mickens and I. Ramadhani, *Journal of Sound and Vibration* (accepted for publication). "Investigation of an anti–symmetric quadratic nonlinear oscillator."

2. R. E. Mickens, *Journal of Sound and Vibration* **111**, 515–518. "A generalization of the method of harmonic balance."

3. J. B. Lewis, *Transactions AIEE*, Pt. II. **72**, 449-453 (1953). "The use of nonlinear feedback to improve the transient response of a servomechanism."

4. P. Hagedorn, *Nonlinear Oscillators* (Clarendon Press, Oxford, 1982); pps. 142, 152, 252–253.

5. R. E. Mickens, *Nonlinear Oscillations* (Cambridge University Press, New York, 1981).

6. F. Verhulst, *Nonlinear Differential Equations and Dynamical Systems* (Springer–Verlag, Berlin, 1990); p. 15.

7. M. L. Boas, *Mathematical Methods in the Physical Sciences* (John Wiley and Sons, New York, 1982; 2nd edition); Chapter 7.

8. R. E. Mickens and M. Mixon, *Journal of Sound and Vibration* (accepted for publication). "Application of generalized harmonic balance to an anti–symmetric quadratic nonlinear oscillator."